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Engineering Notes

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Modified Pseudoinverse Redistribution Methods for Redundant Controls Allocation

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Introduction

IN recent years, control allocation problems have been intensively studied following the work of Durham.[1–3] There are several solution methods: direct control allocation,[1–3] daisy chaining,[4] a linear-programming (LP) method,[5–6] a quadratic-programming (QP) method,[7–11] and a pseudoinverse-redistribution (PIR) method.[10–13] In this Note, a PIR method is discussed.

A pseudoinverse has been used as a selector or a distributor.[14] Shtessel et al. applied a pseudoinverse to the control allocation problem for a tailless aircraft.[15] However, it is fixed, and it does not consider the control surfaces' limits. Virnig and Bodden developed a PIR method for a short takeoff and vertical landing aircraft and then demonstrated the algorithm by means of a simulator.[12] The PIR method repeats the process of calculating a pseudoinverse and a control vector by setting the saturated elements of a control vector to their limit values until a solution is obtained or there is no unsaturated element. Bordignon and Bessolo developed a modified method and applied it to the X-35 aircraft's control allocation problem.[16] Although the PIR method sets the saturated control elements to their limit values, the modified method selects only one saturated element and sets it to its limit value during the redistribution process.

The purpose of this Note is to propose other modified methods and to compare them with the conventional PIR method and Bordignon and Bessolo's method. A different selection of one saturated element results in a difference in performance. For this purpose, two different selections are proposed. The performance metrics of concern are the calculation times, the percentage of times that a method converges to an optimal solution, and the characteristics of the errors between the optimal solutions and the redistributed solutions. Numerical examples are presented for comparisons.

Control Allocation Problem

There are several ways to express a control allocation problem. Here, a QP formulation is followed. A dynamic system with a control input matrix $B \in \Re^{n \times l}$ is assumed to have redundant controls, namely, $l > n$, and the controls are constrained by deflection limits ($u_{\min}$ and $u_{\max}$). For a desired moment m_d, it is desirable for the redundant controls to realize the desired moment, namely, $Bu - m_d = 0$ using the minimum control power as

$$\underset{u}{\text{minimize}} \qquad P_1 = \tfrac{1}{2}u^T u$$

$$\text{subject to} \qquad u_{\min} \le u \le u_{\max}, \qquad Bu - m_d = 0 \tag{1}$$

If the control is unable to satisfy the equality condition, it is desirable to minimize the error as

$$\underset{u}{\text{minimize}} \qquad P_2 = \tfrac{1}{2}(Bu - m_d)^T(Bu - m_d)$$

$$\text{subject to} \qquad u_{\min} \le u \le u_{\max} \tag{2}$$

The desired moment m_d determines whether or not the condition $Bu - m_d = 0$ can be satisfied within the control limits for a given system with fixed B, $u_{\min}$, and $u_{\max}$. If the equality condition can be satisfied, then one attempts to find a minimum norm solution, and if not, one searches for a minimum error solution. Equations (1) and (2) are general formulations of a control allocation problem,[8,11] and the active set method[9,17] is a powerful solution method.

If all of the inequality constraints are inactive, the optimal solution or control vector is obtained by a right pseudoinverse as in the following:

$$u = B^T(BB^T)^{-1}m_d \equiv B^+ m_d \tag{3}$$

However, if the inequality constraints are violated, that is, some elements of u of Eq. (3) exceed the limits, a recalculation is necessary to obtain an optimal solution within the limits. Although optimization algorithms perform recalculations systematically, the PIR method performs recalculations in an ad hoc manner: all of the saturated elements are set to their limit values, and a pseudoinverse and a control vector are recalculated with the unsaturated elements. This is simple and fast when compared with an optimization algorithm, and it can provide optimal solutions in some cases.

Modified Pseudoinverse Redistribution Methods

We define a diagonal matrix E whose diagonal term is $E_{ii} = \max(|u_{\max,i}|, |u_{\min,i}|)$. Then we use the normalized control vector $\hat{u}$ and the control input matrix $\hat{B}$ defined as $\hat{u} = E^{-1}u$ and $\hat{B} = BE$. In this case, $|\hat{u}_{\max,i}|, |\hat{u}_{\min,i}| \le 1$. Normalizing the control vector is the same as considering a weighting matrix in Eq. (1). For convenience, the notation (u, B) will hereafter represent normalized quantities.

First, the PIR method is introduced. If some elements of Eq. (3) exceed their limits, the elements and the control input matrix are separated into the unsaturated and saturated parts as

$$\bar{u} = \begin{bmatrix} u_r \\ u_s \end{bmatrix}, \qquad \bar{B} = [B_r \quad B_s] \tag{4}$$

where u_s are the saturated elements and $\bar{B}$ is the transformed matrix according to $\bar{u}$. Then, u_r is redistributed as follows[12]:

$$u_r = B_r^+(m_d - B_s\tilde{u}_s) \tag{5}$$

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where $\tilde{u}_s$ is the clipped vector of u_s such that the saturated controls are set to their limits. If some elements of u_r exceed their limits, the processes of Eqs. (4) and (5) are repeated until a solution within the limit is obtained or there is no available control left.[12] This PIR method sets all of the saturated elements to their limits at each repeat. This redistribution is simple and fast, but it does not often produce optimal solutions.[11]

Bordignon and Bessolo have proposed a modified redistribution method to use the saturated controls more effectively.[16] The central idea is to set only one saturated control to its limit value at each repeat. At the first iteration, let $\bar{m}_d = m_d$. Among the saturated elements, the one with the largest excess ratio of Eq. (6) is selected and clipped as $\tilde{u}_{s,c}$:

$$k_j = \begin{cases} |u_{s,j}/u_{\max,j}|, & \text{if} \quad u_{s,j} \geq u_{\max,j} \\ |u_{s,j}/u_{\min,j}|, & \text{if} \quad u_{s,j} \leq u_{\min,j} \end{cases} \tag{6}$$

After this element is excluded, B_r is rearranged, and $\bar{m}_d$ is updated as $\bar{m}_d = \bar{m}_d - B_{s,c}\tilde{u}_{s,c}$. The remaining controls are then redistributed as in Eq. (7):

$$u_r = B_r^+ \bar{m}_d \tag{7}$$

$B_{s,c}$ is the column of B_s corresponding to $\tilde{u}_{s,c}$. If u_r is not within the limit, this redistribution process is repeated.

The author proposes two different criteria for selecting the one saturated control. The first one is to select the element that most greatly exceeds its limit. This will be similar to Eq. (6). However, from the second iteration the element signs of Eqs. (3) and (7) are compared. Among the saturated elements whose signs coincide with the original signs in Eq. (3), the one that exceeds its limit the most is selected as $\tilde{u}_{s,c}$. If there is no such element, then the element that exceeds its limit the most is selected:

At the first iteration,

$$|u_{s,j} - u_m|$$

From the second iteration,

$$|u_{s,j} - u_m| \qquad \text{and} \qquad \mathrm{sgn}(u_{s,j}) = \mathrm{sgn}(u_j) \tag{8}$$

where u_j is the element of Eq. (3) corresponding to $u_{s,j}$, and

$$u_m = \begin{cases} u_{\max}, & \text{if} \quad u_{s,j} > u_{\max} \\ u_{\min}, & \text{if} \quad u_{s,j} < u_{\min} \end{cases} \tag{9}$$

The condition of the signs' coincidence is to maintain the direction of the control vector toward the solution of Eq. (3).

The second proposed method is to select the element that reduces the error norm of Eq. (10) the most.

$$\|\bar{m}_d - B_{s,j}\tilde{u}_{s,j}\| \tag{10}$$

We calculate the error norms of $\bar{m}_d - B_{s,j}\tilde{u}_{s,j}$ by setting an element of u_s to its limit value, compare the error norms, and select the element yielding the minimum error norm.

The following is a summary of the modified PIR methods.

Step 1: $\bar{m}_d = m_d$, $B_r = B$, and $\tilde{u}_s =$ null.

Step 2: $u = B_r^+ \bar{m}_d$. If $u_{\min} \leq u \leq u_{\max}$, stop.

Step 3: Separate u as Eq. (4), and select $\tilde{u}_{s,c}$ based on Eq. (6), (8), or (10).

Step 4: Include $\tilde{u}_{s,c}$ into $\tilde{u}_s$, and update B_r and B_s according to $\tilde{u}_s$.

Step 5: If there is no control left, go to step 7.

Step 6: $\bar{m}_d = \bar{m}_d - B_{s,c}\tilde{u}_{s,c}$. Go to step 2.

Step 7: $\bar{u}^T = [u_r^T \;\; \tilde{u}_s^T]$. Rearrange $\bar{u}$ as u, and stop.

During the redistributions, a left pseudoinverse is used at step 2 if the columns of B_r are fewer than the rows of B_r.

Numerical Examples

The performance of the PIR method, the modified PIR methods, and an optimization method are compared by means of several numerical models. The performance metrics of concern are the percentage of times optimal solutions were found, the calculation times, and the statistical measures of the error characteristics. Numerical

Table 1 Comparison of the test results

Model	Case	Performance	ASM[a]	PIR[b]	MPIR-B[c]	MPIR-(8)[d]	MPIR-(10)[e]
1	1	PTOS (%)[f]	——	99.936	99.985	99.985	99.966
		CT[g]	3.16	1.00	1.01	1.02	1.03
		EC[h] (A/SD)[i]	——	0.083/0.070	0.038/0.035	0.038/0.035	0.075/0.054
	2	PTOS (%)	——	59.02	79.71	71.10	97.37
		CT	2.78	1.00	1.73	1.95	1.81
		EC (A/SD)	——	1.254/0.889	1.258/0.708	1.306/0.751	0.206/0.256
2	1	PTOS (%)	——	99.994	99.997	99.997	99.997
		CT	3.35	1.00	1.01	1.02	1.03
		EC (A/SD)	——	0.754/0.516	0.273/0.282	0.273/0.282	0.318/0.247
	2	PTOS (%)	——	37.26	35.68	36.34	67.16
		CT	3.72	1.00	2.01	2.31	2.19
		EC (A/SD)	——	2.272/0.931	2.301/0.836	2.128/0.853	1.027/0.752
3	1	PTOS (%)	——	99.890	99.966	99.999	99.957
		CT	3.15	1.00	1.01	1.03	1.02
		EC (A/SD)	——	0.042/0.039	0.029/0.041	0.079/0	0.036/0.030
	2	PTOS (%)	——	14.18	26.53	30.26	77.67
		CT	4.90	1.00	3.20	3.70	3.59
		EC (A/SD)	——	1.566/0.774	1.776/0.873	1.527/0.775	0.925/0.805
4	1	PTOS (%)	——	99.995	99.993	99.995	99.995
		CT	2.77	1.00	1.03	1.06	1.03
		EC (A/SD)	——	0.033/0.036	0.037/0.038	0.035/0.035	0.021/0.038
	2	PTOS (%)	——	18.00	9.60	26.15	70.38
		CT	9.90	1.00	3.50	3.81	3.83
		EC (A/SD)	——	1.705/0.769	2.411/0.926	2.044/0.856	0.865/0.850

[a]ASM = active set method (optimal method).
[b]PIR = PIR method.
[c]MPIR-B = MPIR method by Eq. (6).
[d]MPIR method by Eq. (8).
[e]MPIR method by Eq. (10).
[f]PTOS = percentage of times optimal solutions were found (%).
[g]CT = calculation times relative to the PIR method.
[h]EC = error characteristics.
[i](A/SD) = (average/standard deviation).

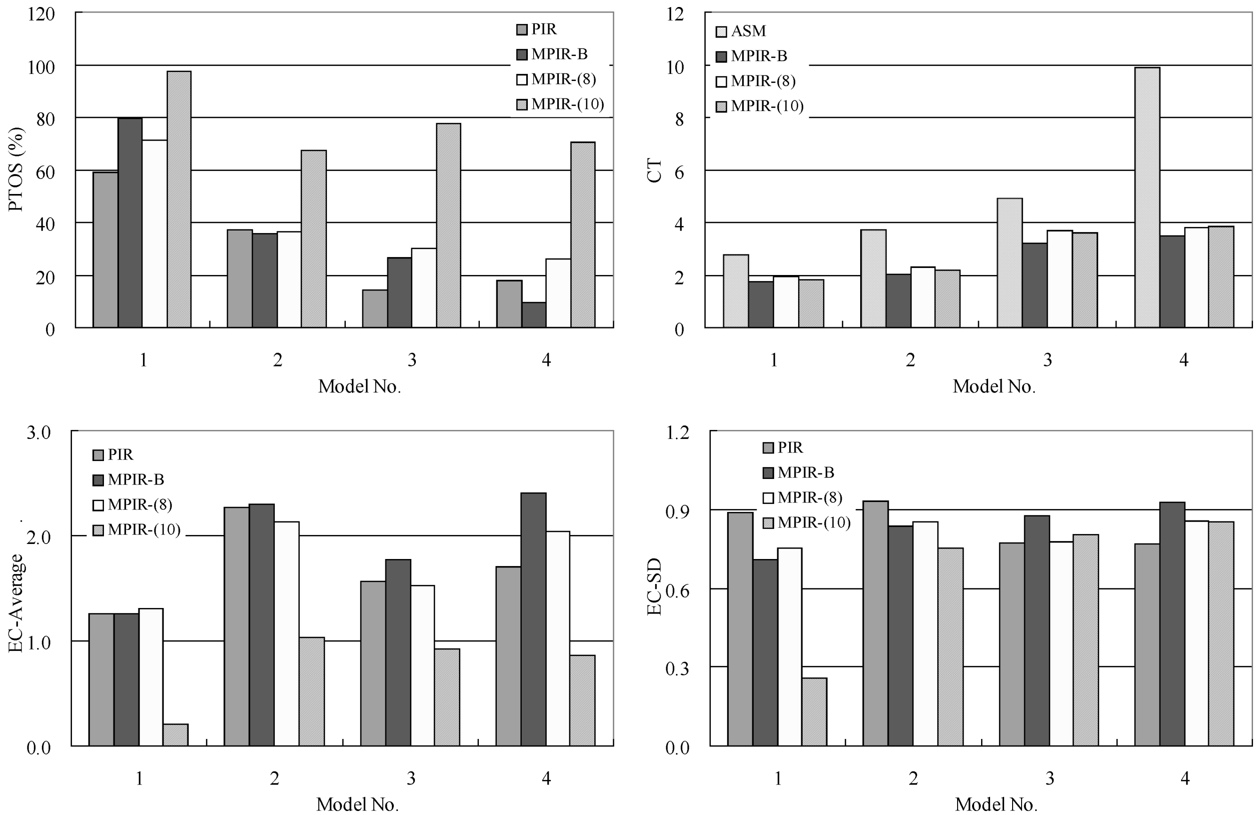

Fig. 1 Comparison of the test results for the unattainable area (case 2).

models selected from several references are as follows (numerical data are listed in the Appendix, and they are not normalized): model 1, Ref. 3, 4 inputs and 3 outputs, a test model; model 2, Ref. 18, 5 inputs and 3 outputs, an aircraft model; model 3, Ref. 19, 10 inputs and 3 outputs, an aircraft model; and model 4, Ref. 5, 12 inputs and 3 outputs, a tailless aircraft model.

Regarding each model, two test sets of 100,000 m_d are randomly generated as follows:

Case 1, attainable area: $m_d = Bu$, $-1 \le u_{\min} \le u \le u_{\max} \le 1$.

Case 2, unattainable area: $m_d = Bu$, $-4 \le u \le 4$. If m_d is attainable, it is excluded.

The five methods solve the same test sets. Because m_d is randomly generated, the results might be different for another test set. However, the difference is insignificant. The test program has been run using MATLAB® on a 3.0-GHz Pentium IV PC. The test results are presented in Table 1. The selected optimization method is the active set method (ASM).[9] The solution obtained by this method is the optimal solution u_{opt}, where u is considered to be optimal if the error norm $\|u - u_{\text{opt}}\|$ is less than 1×10^{-6}. The calculation times are relative values to the PIR method's calculation time. The average and the standard deviation values of the error norms have been used to represent the error characteristics.

For case 1, all of the redistribution methods almost always have found near-optimal solutions, and in those cases where they did not the errors are not excessive. The calculation times of all of the redistribution methods are almost the same. The performance metrics of all of the redistribution methods are satisfactory and independent of the number of control effectors.

Figure 1 is another representation of the test results of case 2. It could be seen that the percentage of times optimal solutions were found by the redistribution methods decreased when compared to case 1. It could also be seen that the MPIR method of Eq. (10) still outperformed the other methods in terms of the number of optimal solutions found. The calculation times of the modified methods have increased because the numbers of iterations increased. The increment of the calculation time depends on the number of control effectors. The MPIR method of Eq. (10) has presented the smallest average of the error norm. The error characteristics are independent of the number of control effectors. For case 2 of model 4, some controls are coplanar during the calculations. Hence, a pseudoinverse has been calculated by the singular value decomposition method.

Conclusions

This Note introduced two new pseudoinverse-redistribu- tion (PIR) methods for control allocation. The proposed methods have been compared with a conventional PIR method as well as Bordignon and Bessolo's modified method by means of numerical examples. Three performance metrics were used to compare the methods: the percentage of times optimal solutions were found, the time required to compute solutions relative to the baseline PIR method, and statistical measures of the error characteristics. For attainable desired moments, all methods converged to the optimal solutions in almost all cases. For unattainable desired moments, one of the modified PIR methods proposed in this Note was found to be superior to the other methods studied, in terms of the percentage of optimal solutions found and in terms of the error characteristics. The improved methods proposed in this Note can be used as direct replacements for conventional PIR methods.

Appendix: Numerical Data for the Selected Systems

Model 1, Ref. 3:

$$B = \begin{bmatrix} 0.2 & -0.2 & 0.8 & 0.1 \\ -0.5 & -0.5 & -1.0 & 0.0 \\ -0.3 & 0.3 & -0.4 & -0.5 \end{bmatrix}$$

$$u_{\max} = -u_{\min} = [1 \quad 1 \quad 1 \quad 1]^T \text{(normalized)}$$

Model 2, Ref. 18:

$$B = \begin{bmatrix} 0.006 & 0.006 & 0.004 & 0 & 0.090 \\ 1.879 & 1.328 & 0.029 & 0.675 & 0.217 \\ -0.109 & -0.096 & -0.084 & 0.007 & -2.974 \end{bmatrix}$$

$$u_{\max} = -u_{\min} = [17.5 \quad 27.5 \quad 30 \quad 30 \quad 30]^T \text{(deg)}$$

Model 3, Ref. 19:

$$B = \begin{bmatrix} -4.38 & 4.38 & -5.84 & 5.84 & 1.67 & -6.28 & 6.28 & 2.92 & 0.001 & 1.0 \\ -53.3 & -53.3 & -6.49 & -6.49 & 0 & 6.23 & 6.23 & 0.001 & 35.53 & 0.001 \\ 1.1 & -1.1 & 3.91 & -3.91 & -7.43 & 0 & 0 & 0.03 & 0.001 & 14.85 \end{bmatrix} \times 10^{-2}$$

$$u_{\max} = [0.183 \quad 0.183 \quad 0.524 \quad 0.524 \quad 0.524 \quad 0.785 \quad 0.785 \quad 0.524 \quad 0.524 \quad 0.524]^T \,(\text{rad})$$

$$u_{\min} = -[0.419 \quad 0.419 \quad 0.524 \quad 0.524 \quad 0.524 \quad 0.140 \quad 0.140 \quad 0.524 \quad 0.524 \quad 0.524]^T \,(\text{rad})$$

Model 4, Ref. 5:

$$B = \begin{bmatrix} 34 & -34 & 53 & -53 & -0.1 & 0.1 & 3.7 & -3.7 & -92 & 681 & 0 & -0.1 \\ -5.5 & -5.5 & -14 & -14 & 0 & 0 & 14 & 14 & 0 & 0 & -3.9 & 0 \\ -0.2 & 0.2 & 1.6 & -1.6 & -4.2 & 4.2 & 3 & -3 & -2.4 & 29012 & 0 & -3.4 \end{bmatrix}$$

$$u_{\max} = [30 \quad 30 \quad 45 \quad 45 \quad 90 \quad 90 \quad 10 \quad 10 \quad 1 \quad 0.006 \quad 30 \quad 30]^T \,(\text{deg except element10})$$

$$u_{\min} = -[30 \quad 30 \quad 30 \quad 30 \quad 0 \quad 0 \quad 80 \quad 80 \quad 1 \quad 0.006 \quad 30 \quad 30]^T$$

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